Unconstrained methods for nonsmooth nonlinear complementarity problems

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ABSTRACT. We consider a nonsmooth nonlinear complementarity problem when the underlying functions admit the H-differentiability but not necessarily locally Lipschitzian nor directionally differentiable. We study the connection between the solutions of the nonsmooth nonlinear complementarity problem and global/local/stationary points of the associated square penalized Fischer-Burmeister and square Kanzow-Kleinmichel merit functions. We show under appropriate regularity conditions on an *H*-differential of f minimizing a merit function corresponding to f leads to a solution of the nonlinear complementarity problem.

KEYWORDS: _H-Differentiability, Locally Lipschitzian, Merit function, NCP function, Nonlinear complementarity problem, Semismooth-functions, Unconstrained minimization_

Consider the nonsmooth nonlinear complementarity problem, denoted by the NCP(*f*),which is to find a vector in R^n satisfying the conditions

$$\bar{x} \in R^n \text{such that } \bar{x} \geq 0, f(\bar{x}) \geq 0 \text{ and} \left\langle f(\bar{x}), \bar{x} \right\rangle = 0$$

where $f : R^n \rightarrow R^n R^m$ is a given *H*-differentiable function not necessarily locally Lipschitzian nor directionally differentiable. Nonlinear complementarity problem arises in many applications, e.g., in operations research, economic equilibrium models and engineering sciences (contact problems, obstacle problems, equilibrium models,...) (for a more detail description see Ferris, Pang, 1997; Harker, Pang, 1990). Also, NCP has been served as a general framework for linear, quadratic, and nonlinear programming. When f is a continuously differentiable or locally Lipschitzian, one of the well-known approaches to solve the NCP is to reformulate the original NCP as an unconstrained minimization problem whose global minima are coincident with the solution of the NCP and the objective function of this unconstrained minimization problem is called a merit function for the NCP, (Facchinei, Kanzow, 1997; Facchinei, Soares, 1997; Fischer, 1998 and 1997; Fischer et al., 2001; Geiger, Kanzow, 1996; Jiang, 1996; Kanzow, 1996; De Luca et al., 1996; Mangasarian, Solodov, 1993; Yamashita, Fukushima, 1995). Most of the merit functions in these references based on the square Fischer-Burmeister function (Facchinei, Soares, 1997; Fischer, 1998; Geiger, Kanzow, 1996; Jiang, 1996; Kanzow, 1996; De Luca et al., 1996) the implicit Lagrangian function (Jiang, 1996; Mangasarian, Solodov, 1993; Yamashita, Fukushima, 1995) and for other NCP functions see, e.g., the survey paper (Fischer, 1995). Most of these methods rely on the a so-called NCP function: An NCP function is a function $\varphi : R^2 \rightarrow R$ having the following property

$$\phi(a,b) = 0 \Leftrightarrow ab = 0, a \geq 0, b \geq 0$$

For the problem NCP(f), we define $\Phi : R^n \rightarrow R^n$ by

$$\Phi(x) = \begin{bmatrix} \phi(x_1, f_1(x)) \\ \vdots \\ \phi(x_i, f_i(x)) \\ \vdots \\ \phi(x_n, f_n(x)) \end{bmatrix}, \tag{1}$$

then it follows immediately from the definition of an NCP function that

$$\bar{x} \text{ solves NCP } (f) \quad \Leftrightarrow \quad \Phi(\bar{x}) = 0 \quad \Leftrightarrow \quad \Psi(\bar{x}) = 0$$

where $\Psi : R^n \rightarrow R$ denotes the corresponding merit function

$$\Psi(x) := \sum_{i=1}^{n} \Phi_i(x). \tag{2}$$

By abuse of language, we call $\Phi(x)$ an NCP function for NCP(f). In this paper, we consider the following NCP functions:

(1)

$$\phi_{SPFB}(a,b) = \frac{1}{2}[\phi_\lambda(a,b)]^2 := \frac{1}{2}[\lambda\phi_{FB}(a,b) + (1-\lambda)a_+b_+]^2 \tag{3}$$

where ϕ_{SPFB}, $\phi_\lambda : R^2 \rightarrow R$. NCP function ϕ_λ is called the penalized Fischer-Burmeister function (Chen et al., 2000)

$$\phi_{\lambda}(a,b) := \lambda\phi_{FB}(a,b) + (1-\lambda)a_+b_+ \tag{4}$$

where ϕ_{FB} is called Fischer-Burmeister function, $a_+ = \max\{0, a\}$ and $\lambda \in (0,1)$ is a fixed parameter. Then its merit function associated to ϕ_{SPFB} at $\bar{x}$ is defined as in (2) where

$$\begin{aligned}\Phi_i(\bar{x}) &= \phi_{SPFB}(\bar{x}_i, f_i(\bar{x})) = \frac{1}{2}\left[\phi_{\lambda}(\bar{x}_i, f_i(\bar{x}))\right]^2 \\ &:= \frac{1}{2}\left[\lambda\phi_{FB}(\bar{x}_i, f_i(\bar{x})) + (1-\lambda)\bar{x}_{i+} f_i(\bar{x})_+\right]^2. \end{aligned} \tag{5}$$

(2)

$$\phi_{SKK}(a,b) := \frac{1}{2}\left[\phi_{\beta}(a,b)\right]^2 = \frac{1}{2}\left[a+b-\sqrt{(a-b)^2+\beta ab}\right]^2 \tag{6}$$

where $\phi_{SKK}; \phi_{\beta} : R^2 \rightarrow R$. NCP function ϕ_{β} was proposed by Kanzow-Kleinmichel (Kanzow, Kleinmichel, 1998)

$$\phi_{\beta}(a,b) := a+b-\sqrt{(a-b)^2+\beta ab} \tag{7}$$

where β is a fixed parameter in $(0, 4)$. We note that when $\beta = 2$, ϕ reduces to the Fischer-Burmeister function, while as $\beta \rightarrow 0$, ϕ_{β} becomes

$$\phi(a,b) := a+b-\sqrt{(a-b)^2} \quad (=2\min\{a,b\}).$$

Then the merit function associated to ϕ_{SKK} at $\bar{x}$ is defined as in (2) where

$$\begin{aligned}\Phi_i(\bar{x}) &= \phi_{SKK}\left(\bar{x}_i, f_i(\bar{x})\right) = \frac{1}{2}\left[\phi_{\beta}(\bar{x}_i, f_i(\bar{x}))\right]^2 \\ &:= (1/2)\left[\bar{x}_i + f_i(\bar{x}) - \sqrt{\left(\bar{x}_i - f_i(\bar{x})\right)^2 + \beta\bar{x}_i f_i(\bar{x})}\right]^2. \end{aligned} \tag{8}$$

The main goal of this paper is to study the connection between the solutions of the nonsmooth nonlinear complementarity problem and global/local/stationary points of the associated square penalized Fischer-Burmeister and square Kanzow-Kleinmichel merit functions. The organization of the paper is as follows. Section 2, we state some basic definitions and preliminary results. In Section 3, we describe *H*-differentials of the square Kanzow-Kleinmichel function and the square penalized Fischer-Burmeister function, and their merit functions. Also, we show how, under appropriate regularity -conditions on an *H*-differential of *f*, finding local/global minimum of Ψ (or a "stationary point" of Ψ) leads to a solution of the given nonlinear complementarity problem. Our results unify/extend various similar results proved in the literature for C^1 and semismooth functions (Chen et al., 2000; Kanzow, Kleinmichel, 1998).

Preliminaries

A few words about notation. We regard vectors in R^n as column vectors. We denote the inner-product between two vectors x and y in R^n by either $x^T y$ or $\langle x, y\rangle$. Vector inequalities are interpreted componentwise. A subscript i is used to denote ith component of a vector $x \in R^n$. A superscript k indicates the kth iterate of a given sequence. For a matrix A, A_i denotes the ith row of A. For a differentiable function $f : R^n \rightarrow R^m$, $\nabla f(\bar{x})$ denotes the Jacobian matrix of f at $\bar{x}$. We call ϕ a nonnegative NCP function if $\phi(a,b) \geq 0$ on R^2. We call Φ a nonnegative NCP function for NCP(f) if ϕ is nonnegative. We need the following definitions from (Cottle et al., 1992; Moré et al., 1973).

Definition 2.1 *A matrix* $A \in R^{n\times n}$ is called $\boldsymbol{P}_0$ ($\boldsymbol{P}$) if $\forall x \in R^n, x \neq 0$, *there exists i such that* $x_i \neq 0$ *and* $x_i (Ax)_i \geq 0$ (>0) *or equivalently, every principle minor of* A *is nonnegative (respectively, positive).*

Definition 2.2 *For a function* $f : R^n \rightarrow R^n$, *we say that* f *is a*
(i) *monotone if*

$$\langle f(x) - f(y), x - y\rangle \geq 0 \quad \text{for all } x, y \in R^n.$$

(ii) $\boldsymbol{P_0}$ ($\boldsymbol{P}$) -function if, for any $x \neq y$ *in* R^n,

$$\max_{\{i: x_i \neq y_i\}} (x-y)_i [f(x) - f(y)]_i \geq 0 \, (>0). \tag{9}$$

We note that every monotone (strictly monotone) function is a $\boldsymbol{P}_0$ ($\boldsymbol{P}$)-function.
The following result is from (Moré et al., 1973; Song et al., 2000).

Theorem 2.1 *Under each the following conditions,* $f : R^n \rightarrow R^n$ is a $\boldsymbol{P}_0$ ($\boldsymbol{P}$)*-function.*

(a) *f is Fréchet differentiable on R^n and for every $x \in R^n$, the Jacobian matrix $\nabla f(x)$ is a $\boldsymbol{P}_0$ ($\boldsymbol{P}$)-matrix.*
(b) *f is locally Lipschitzian on R^n and for every $x \in R^n$, the generalized Jacobian $\partial f(x)$ consists of $\boldsymbol{P}_0$ ($\boldsymbol{P}$)-matrices.*
(c) *f is semismooth on R^n (in particular, piecewise affine or piecewise smooth) and for every $x \in R^n$, the Bouligand subdifferential $\partial_B f(x)$ consists of $\boldsymbol{P}_0$ ($\boldsymbol{P}$)-matrices.*
(d) *f is H-differentiable on R^n and for every $x \in R^n$, an H-differential $T_f(x)$ consists of $\boldsymbol{P}_0$ ($\boldsymbol{P}$)-matrices.*

We give the following definition and examples from Gowda and Ravindran (2000).

Definition 2.3 *Given a function $f : \Omega \subseteq R^n \rightarrow R^m$ where Ω is an open set in R^n and $x^* \in \Omega$, we say that a nonempty subset $T(x^*)$ (also denoted by $T_f(\bar{x}^*)$) of $R^{m\times n}$ is an H-differential of f at x^* if for every sequence $\{x^k\} \subseteq \Omega$ converging to x^*, there exist a subsequence $\{x^{k_j}\}$ and a matrix $A \in T(x^*)$ such that*

$$f\left(x^{k_j}\right) - f\left(x^*\right) - A\left(x^{k_j} - x^*\right) = o\left(\left\|x_j^k - x^*\right\|\right). \qquad (10)$$

We say that f is H-differentiable at x^* if f has an H-differential at x^*.

Remarks

As noted in (Tawhid, Gowda, 2000), it is easily seen that if a function $f : \Omega \subseteq R^n \rightarrow R^m$ is H-differentiable at a point $\bar{x}$, then there exist a constant $L > 0$ and a neighbourhood $B(\bar{x};\, \delta)$ of x with

$$\left\|f(x) - f(\bar{x})\right\| \le L\left\|x - \bar{x}\right\|, \ \forall x \in B(\bar{x}, \delta). \qquad (11)$$

Conversely, if condition (11) holds, then $T(\bar{x}) := R^{m\times n}$ can be taken as an *H*-differential of f at $\bar{x}$. We thus have, in (11), an alternate description of *H*-differentiability. But, as we see in the sequel, it is the identification of an appropriate *H*-differential that becomes important and relevant. Clearly any function locally Lipschitzian at $\bar{x}$ will satisfy (11). For real valued functions, condition (11) is known as the `calmness' of f at $\bar{x}$. This concept has been well studied in the literature of nonsmooth analysis (Rockafellar et al., 1998, Chapter 8).

Example 1 Let $f : R^n \to R^m$ be Fréchet differentiable at $x^* \in R^n$ with Fréchet derivative matrix (= Jacobian matrix derivative) $\{\nabla f(x^*)\}$ such that

$$f(x) - f(x^*) - \nabla f(x^*)(x - x^*) = o(\|x - x^*\|).$$

Then f is *H*-differentiable with $\{\nabla f(x^*)\}$ as an *H*-differential.

Example 2 Let $f : \Omega \subseteq R^n \to R^m$ be locally Lipschitzian at each point of an open set Ω.
For $x^* \in \Omega$, define the Bouligand subdifferential of f at x^* by

$$\partial_B f(x^*) = \left\{\lim \nabla f(x^k) : x^k \to x^*, x^k \in \Omega_F\right\}$$

where Ω_f is the set of all points in Ω where f is Frèchet differentiable. Then, the (Clarke) generalized Jacobian (Clarke, 1983)

$$\partial f(x^*) = co\partial_B f(x^*)$$

is an *H*-differential of f at x^*.

Example 3 Consider a locally Lipschitzian function $f : \Omega \subseteq R^n \to R^m$ that is semismooth at $x^* \in \Omega$ (Miffin, 1977; Qi, 1993; Qi, Sun, 1993). This means for any sequence $x^k \to x^*$, and for $V_k \in \partial f(x^k)$,

$$f(x^k) - f(x^*) - V_k(x^k - x^*) = o(\|x^k - x^*\|).$$

Then the Bouligand subdifferential

$$\partial_B f(x^*) = \left\{\lim \nabla f(x^k) : x^k \to x^*, x^k \in \Omega_f\right\}.$$

is an *H*-differential of f at x^*. In particular, this holds if f is piecewise smooth, i.e., there exist continuously differentiable functions $f_j : R^n \to R^m$ such that

$$f(x) \in \{f_1(x), f_2(x), \dots, f_J(x)\} \quad \forall x \in R^n.$$

Example 4 Let $f: R^n \rightarrow R^n$ be *C*-differentiable (Qi, 1996) in a neighborhood D *of* x^*. This means that there is a compact upper semicontinuous multivalued mapping $x \mapsto T(x)$ with $x \in D$ and $T(x) \subset R^{n\times n}$ satisfying the following condition at any $a \in D$: *For* $V \in T(x)$,

$$f(x) - f(a) - V(x-a) = o(\|x-a\|)$$

Then, f is *H*-differentiable at x^* with $T(x^*)$ as an *H*-differential.

Remark While the Fréchet derivative of a differentiable function, the Clarke generalized Jacobian of a locally Lipschitzian function (Clarke, 1986), the Bouligand differential of a semismooth function (Qi, 1993), and the *C*-differential of a *C*-differentiable function (Qi, 1996) are particular instances of *H*-differential, the following simple example, is taken from (Gowda, 1998), shows that an *H*-differentiable function need not be locally Lipschitzian nor directionally differentiable.

Example 5 Consider on R,

$$f(x) = x\sin(\frac{1}{x}) \quad \text{for } x \neq 0 \text{ and } f(0) = 0.$$

Then f is *H*-differentiable on R with

$$T(0) = [-1,1] \text{ and } T(c) = \left\{ \sin\left(\frac{1}{c}\right) - \frac{1}{c}\cos\left(\frac{1}{c}\right) \right\} \quad \text{for } c \neq 0.$$

We note that f is not locally Lipschitzian around zero. We also see that f is neither Fréchet differentiable nor directionally differentiable.

The main results

Before stating our results, we would like to mention that we employ the concepts of *H*-differentiability and *H*-differential of a function (Gowda, 2000) due to the following reasons: the Fréchet derivative of a differentiable function, the Clarke generalized Jacobian of a locally Lipschitzian function (Clarke, 1983), the Bouligand differential of a semismooth function (Qi, 1993), and the *C*-differential of Qi (1996) are particular instances of *H*-differential; any superset of an *H*-differential is an *H*-differential; a *H*-differentiable function need not be locally Lipschitzian function nor directionally differentiable; *H*-differentials enjoy simple sum, product, chain rules, a mean value theorem and a second order Taylor-like expansion, and inverse and implicit function theorems, and *H*-differentiability implies continuity, (Geiger, Kanzow, 1996; Gowda, 1998 and 2004; Gowda, Ravindran, 2000); moreover, the closure of the *H*-differential is an approximate Jacobian (Jeyakumar, Luc, 1998).

For some applications of *H*-differentiability to optimization problems, nonlinear complementarity problems and variational inequalities (Tawhid, Gowda, 2000; Tawhid, Goffin, 2008; Tawhid, 2002).

For a given *H*-differentiable function $f : R^n \to R^n$, consider the associated NCP function Φ and the corresponding merit function $\Psi := \Sigma_{i=1}^n \Phi_i$ (as in Examples 6-7 below). It should be recalled that

$$\Psi(\bar{x}) = 0 \Leftrightarrow \Phi(\bar{x}) = 0 \Leftrightarrow \bar{x} \text{ solves NCP}(f).$$

H-differentials of some NCP/merit functions

In this subsection, we compute the *H*-differential of the merit function .

Theorem 3.1 *Suppose* Φ *is H-differentiable at* $\bar{x}$ *with* $T_\Phi(\bar{x})$ *as an H-differential. Then* $\Psi := \Sigma_{i=1}^n \Phi_i$ *is H-differentiable at* $\bar{x}$ *with an H-differential given by*

$$T_\Psi(\bar{x}) = \left\{ e^T B : B \in T_\Phi(\bar{x}) \right\}.$$

Proof. To describe an *H*-differential of Ψ, let $\theta(x)\ \theta = x_1 + \cdots + x_n$. Then $\Psi = \theta \circ \Phi$ so that by the chain rule for *H*-differentiability, we have $T_\Psi(\bar{x}) = (T_\theta \circ T_\Phi)(\bar{x})$ as an *H*-differential of Ψ at $\bar{x}$. Since $T_\theta(\bar{x}) = \{e^T\}$ where e is the vector of ones in R^n, we have

$$T_\Psi(\bar{x}) = \left\{ e^T B : B \in T_\Phi(\bar{x}) \right\}.$$

This completes the proof.

Now we describe the *H*-differentials of the merit functions associated to square Kanzow-Kleinmichel function and square penalized Fischer-Burmeister function.

Example 6 (square Kanzow-Kleinmichel function)
Suppose $f : R^n \to R^n$ has an *H*-differential $T(\bar{x})$ at $\bar{x} \in R^n$. Consider the associated square Kanzow-Kleinmichel function

$$\Phi(\bar{x}) := (1/2)\left[\bar{x} + f(\bar{x}) - \sqrt{(\bar{x} - f(\bar{x}))^2 + \beta \bar{x} f(\bar{x})}\right]^2. \quad (12)$$

where all the operations are performed componentwise. Let

$$J(\bar{x}) = \{i : f_i(\bar{x}) = 0 = \bar{x}_i\}.$$

Then the *H*-differential of Φ in (12) is given by

$$T_\Phi(\bar{x}) = \{VA + W : (A, V, W, d) \in \Gamma\},$$

where Γ is the set of all quadruples (A, V, W, d) with $A \in T(\bar{x})$, $\|d\| = 1$, $V = diag(v_i)$ $W = diag(w_i)$ are diagonal matrices with

$$v_i = \begin{cases} \phi_\beta(\bar{x}_i, f_i(\bar{x}))\left[1 - \dfrac{-2(\bar{x}_i - f_i(\bar{x})) + \beta \bar{x}_i}{2\sqrt{(\bar{x}_i - f_i(\bar{x}))^2 + \beta \bar{x}_i f_i(\bar{x})}}\right] & \text{when } i \notin J(\bar{x}) \\ \phi_\beta(d_i, A_i d)\left[1 - \dfrac{-2(d_i - A_i d) + \beta d_i}{2\sqrt{(d_i - A_i d)^2 + \beta d_i (A_i d)}}\right] & \text{when } i \in J(\bar{x}) \text{ and } (d_i - A_i d)^2 + \beta d_i (A_i d) > 0 \\ \text{arbitrary} & \text{when } i \in J(\bar{x}) \text{ and } (d_i - A_i d)^2 + \beta d_i (A_i d) = 0, \end{cases}$$

$$(13)$$

$$w_i = \begin{cases} \phi_\beta(\bar{x}_i, f_i(\bar{x}))\left[1 - \dfrac{2(\bar{x}_i - f_i(\bar{x})) + \beta f_i(\bar{x})}{2\sqrt{(\bar{x}_i - f_i(\bar{x}))^2 + \beta \bar{x}_i f_i(\bar{x})}}\right] & \text{when } i \notin J(\bar{x}) \\ \phi_\beta(d_i, A_i d)\left[1 - \dfrac{2(d_i - A_i d) + \beta A_i d}{2\sqrt{(d_i - A_i d)^2 + \beta d_i (A_i d)}}\right] & \text{when } i \in J(\bar{x}) \text{ and } (d_i - A_i d)^2 + \beta d_i (A_i d) > 0 \\ \text{arbitrary} & \text{when } i \in J(\bar{x}) \text{ and } (d_i - A_i d)^2 + \beta d_i (A_i d) = 0. \end{cases}$$

We can describe the *H*-differential of Φ in a way similar to the calculation and analysis of Examples 5-7 (Tawhid, Gowda, 2000).

By Theorem 3.1, the *H*-differential $T_\Psi(\bar{x})$ of $\Psi(\bar{x})$ consists of all vectors of the form $v^T A + w^T$ with $A \in T(\bar{x})$, v and w are columns vectors with entries defined by (13).

Example 7 (square penalized Fischer-Burmeister function)
Suppose $f : R^n \to R^n$ has an *H*-differential $T(\bar{x})$ at $\bar{x} \in R^n$. Consider the associated square penalized Fischer-Burmeister function

$$\Phi(\bar{x}) := \frac{1}{2}\left[\lambda\phi_{FB}(\bar{x}, f(\bar{x})) + (1-\lambda)\bar{x}_+ f(\bar{x})_+\right]^2. \qquad (14)$$

where Φ_{FB} is called Fischer-Burmeister function, $a_+ = \max\{0,a\}$ and $\lambda \in (0,1)$ is a fixed parameter, and all the operations are performed componentwise. Let

$$J(\bar{x}) = \{i : f_i(\bar{x}) = 0 = \bar{x}_i\} \text{ and } K(\bar{x}) = \{i : \bar{x}_i > 0, f_i(\bar{x}) > 0\}.$$

For Φ in (14) straightforward calculation shows that an *H*-differential is given by

$$T_\Phi(\bar{x}) = \{VA + W : (A, V, W, d) \in \Gamma\},$$

Where Γ is the set of all quadruples (A, V, W, d) with, $A \in T(\bar{x})$, $\|d\| = 1$, $V = diag(v_i)$ and $W = diag(w_i)$ are diagonal matrices with

$$v_i = \begin{cases} \phi_\lambda(\bar{x}_i, f_i(\bar{x}))\left[\lambda\left(1 - \dfrac{f_i(\bar{x})}{\sqrt{\bar{x}_i^2 + f_i(\bar{x})^2}}\right) + (1-\lambda)\bar{x}_i\right] \text{when } i \in K(\bar{x}) \\ \phi_\lambda(d_i, A_i d)\left[\lambda\left(1 - \dfrac{A_i d}{\sqrt{d_i^2 + (A_i d)^2}}\right)\right] \text{when } i \in J(\bar{x}) \text{ and } d_i^2 + (A_i d)^2 > 0 \\ \phi_\lambda(\bar{x}_i, f_i(\bar{x}))\left[\lambda\left(1 - \dfrac{f_i(\bar{x})}{\sqrt{\bar{x}_i^2 + f_i(\bar{x})^2}}\right)\right] \text{when } i \notin J(\bar{x}) \cup K(\bar{x}) \\ \text{arbitrary} \qquad \text{when } i \in J(\bar{x}) \text{ and } d_i^2 + (A_i d)^2 = 0, \end{cases}$$

(15)

$$w_i = \begin{cases} \phi_\lambda(\bar{x}_i, f_i(\bar{x}))\left[\lambda\left(1-\dfrac{\bar{x}_i}{\sqrt{x_i^2+f_i(\bar{x})^2}}\right)+(1-\lambda)f_i(\bar{x})\right] \text{when } i\in K(\bar{x}) \\ \phi_\lambda(d_i, A_i d)\left[\lambda\left(1-\dfrac{d_i}{\sqrt{d_i^2+(A_i d)^2}}\right)\right] \text{when } i\in J(\bar{x}) \text{ and } d_i^2+(A_i d)^2>0 \\ \phi_\lambda(\bar{x}_i, f_i(\bar{x}))\left[\lambda\left(1-\dfrac{\bar{x}_i}{\sqrt{\bar{x}_i^2+f_i(\bar{x})^2}}\right)\right] \text{when } i\notin J(\bar{x})\cup K(\bar{x}) \\ \text{arbitrary} \qquad \text{when } i\in J(\bar{x}) \text{ and } d_i^2+(A_i d)^2=0. \end{cases}$$

The above calculation relies on the observation that the following is an *H*-differential of the one variable function $z \mapsto z_+$ at any $\bar{z}$:

$$\Delta(\bar{z}) = \begin{cases} \{1\} & \text{if } \bar{z}>0 \\ \{0,1\} & \text{if } \bar{z}=0 \\ \{0\} & \text{if } \bar{z}<0. \end{cases}$$

Using Theorem 3.1, the *H*-differential $T_\Psi(\bar{x})$ of $\Psi(\bar{x})$ consists of all vectors of the form $v^T + A\, w^T$ with $A \in T(\bar{x})$, v and w are columns vectors with entries defined by (15).

We close this subsection by the following lemma that will be needed in the sequel. The proof is similar to lemmas 1-5 in (Geiger, Kanzow, 1996).

Lemma 3.1 *Assume that Ψ is H-differentiable with an H-differential $T_\Psi(\bar{x})$ and Φ (as in Examples 6-7) is nonnegative H-differentiable with an H-differential $T_\Phi(\bar{x})$ is given by*

$$T_\Phi(\bar{x}) = \{VA+W : A\in T(\bar{x}), V = diag(v_i) \text{ and } W = diag(w_i)\} \qquad (16)$$

where Φ, V and W satisfy the following properties:

$$\left.\begin{array}{ll} \text{(i)} & \bar{x} \textit{ solves } \mathrm{NCP}(f) \Leftrightarrow \Phi(\bar{x})=0. \\ \text{(ii)} & \textit{For } i \in \{1,\dots,n\}, v_i w_i \geq 0. \\ \text{(iii)} & \textit{For } i\in\{1,\dots,n\}, \Phi_i(\bar{x})=0 \Leftrightarrow (v_i,w_i)=(0,0). \\ \text{(iv)} & \textit{For } i \in\{1,\dots,n\}, \textit{with } \bar{x}_i\geq 0 \text{ and } f(\bar{x}_i)\geq 0, \textit{ we have } v_i \geq 0. \\ \text{(v)} & \textit{If } 0\in T_\Psi(\bar{x}), \textit{then } \Phi(\bar{x})=0 \Leftrightarrow v=0. \end{array}\right\} \qquad (17)$$

Minimizing the merit function under regularity (strict regularity) conditions

We generalize the concept of a regular (strictly regular) point from (Facchinei, Kanzow, 1997; Ferris, Ralph, 1995; De Luca et al., 1996; Moré, 1996). For a given *H*-differentiable function f and $x \in R^n$, we define the following index sets:

$$\mathcal{P}(\bar{x}) := \{i : v_i > 0\}, \quad \mathcal{N}(\bar{x}) := \{i : v_i < 0\},$$
$$\mathcal{C}(\bar{x}) := \{i : v_i = 0\}, \quad \mathcal{R}(\bar{x}) := \mathcal{P}(\bar{x}) \cup \mathcal{N}(x)$$

where v_i are the entries of V in (16) (e.g., v_i is defined as in Examples 6-7).

Definition 3.1 *Consider f, Φ, and Ψ as in Examples 6-7. A vector $x^* \in R^n$ is called strictly regular if, for every nonzero vector $z \in R^n$ such that*

$$z_{\mathcal{C}} = 0, \;\; z_{\mathcal{P}} > 0, z_{\mathcal{N}} < 0, \tag{18}$$

there exists a vector $\mathrm{s} \in \mathrm{R}^n$ such that

$$s_{\mathcal{P}} \geq 0, \;\; s_{\mathcal{N}} \leq 0, s_{\mathcal{C}} = 0, \;\; and \tag{19}$$
$$s^T \mathrm{A}^T z \; < \; 0, \quad for \; all \; A \in T(x^*). \tag{20}$$

Theorem 3.2 *Suppose $f : R^n \rightarrow R^n$ is H-differentiable at $\bar{x}$ with an H-differential $T(\bar{x})$. Let Φ be as in Examples 6-7. Assume that $\Psi := \sum_{i=1}^{n} \Phi_i(\bar{x})$ is H-differentiable at $\bar{x}$* with *an H-differential given by*

$$T_\Psi(\bar{x}) = \left\{ v^T A + w^T : (A, v, w) \in \Omega \right\}$$

where Ω is the set all triples (A, v, w) with A∈T($\bar{x}$), v and w vectors in R^n satisfying properties (ii), (iii), and (v) in (17).
Then $\bar{x}$ solves NCP(f) if and only if $0 \in T_\Psi(\bar{x})$ and $\bar{x}$ is a strictly regular point.

Proof. The "if" part of the theorem follows easily from the definitions. Now suppose that $0 \in T_\Psi(\bar{x})$ and $\bar{x}$ is a strictly regular point. Then for some $v^T A + w^T \in T_\Psi(\bar{x})$,

$$0 = v^T A + w^T \Rightarrow A^T v + w = 0. \tag{21}$$

We claim that $\Phi(\bar{x}) = 0$. Assume the contrary that $\bar{x}$ is not a solution of NCP(f). Then by property (v) in (17), we have w as a nonzero vector satisfying $v_{\mathcal{C}} = 0, v_{\mathcal{P}} > 0; \ v_{\mathcal{N}} < 0.$
Since $\bar{x}$ is a strictly regular point, and $v_i w_i \geq 0$ by property (ii) in (17), by taking a vector $s \in R^n$ satisfying (19) and (20), we have

$$s^T A^T v \geq 0 \tag{22}$$

and

$$s^T w = s_{\mathcal{C}}^T w_{\mathcal{C}} + s_{\mathcal{P}}^T w_{\mathcal{P}} + s_{\mathcal{N}}^T w_{\mathcal{N}} \geq 0 \tag{23}$$

Thus we have $s^T (A^T v + w) = s^T A^T v + s^T w > 0$. We reach a contradiction to (21). Hence, $\bar{x}$ is a solution of NCP(f).
Now we state a consequence of the above theorem.

Theorem 3.3 *Suppose $f : R^n \to R^n$ is H-differentiable at $\bar{x}$ with an H-differential $T(\bar{x})$. Let Φ be as in Examples 6-7. Assume that $\Psi := \Sigma_{i=1}^{n} \Phi_i(\bar{x})$ is H-d*
ifferentiable at $\bar{x}$ with an H-differential given by

$$T_\Psi(\bar{x}) = \left\{ v^T A + w^T : (A, v, w) \in \Omega \right\}$$

where Ω is the set all triples (A, v, w) with $A \in T(\bar{x})$, v and w vectors in R^n satisfying properties (ii), (iii), and (v) in (17).
Further suppose that $T(\bar{x})$ consists of positive-definite matrices. Then

$$\Phi(\bar{x}) = 0 \Leftrightarrow 0 \in T_\Psi(\bar{x}).$$

Proof. The proof follows by taking $s = z$ in Definition 3.1 of a strictly regular point andby using Theorem 3.2.
Before we state the next theorem, we recall a definition from (Song et al., 1999).

Definition 3.2 *Consider a nonempty set C in $R^{n \times n}$. We say that a matrix A is a row representative of C if for each index $i = 1, 2, \ldots, n$, the ith row of A is the ith row of some matrix $C \in$ C. We say that C has the row-$\boldsymbol{P}_0$-property (row-$\boldsymbol{P}$-property) if every row representative of C is a $\boldsymbol{P}_0$-matrix ($\boldsymbol{P}$-matrix). We say that C has* *the column-P_0-property (column-P-property) if $C^T = \{A^T : A \in C\}$ has the row-P_0-property (row-P-property).*

Theorem 3.4 *Suppose $f : R^n \rightarrow R^n$ is H-differentiable at $\bar{x}$ with an H-differential $T(\bar{x})$. Let Φ be as in Examples 6-7. Assume that $\Psi := \Sigma_{i=1}^{n} \Phi_i(\bar{x})$ is H-differentiable at $\bar{x}$ with an H-differential given by*

$$T_\Psi(\bar{x}) = \left\{ v^T A + w^T : (A, v, w) \in \Omega \right\}.$$

where Ω is the set all triples (A, v, w) with $A \in T(\bar{x}), v$ and w vectors in R^n satisfying properties (ii), (iii), and (v) in (17). Furtner suppose that $T(\bar{x})$ has the column-P-property. Then

$$\bar{x} \text{ solves NCP}(f) \text{ if and only if } 0 \in T_\Psi(\bar{x}).$$

Proof. In view of Theorem 3.v., it is enough to show $\bar{x}$ is a strictly regular point. To see this, let be a nonzero vector satisfying (18). Since $T(\bar{x})$ has the column-P-property, by Theorem 2 in Song et al., (1999), there exists an index j such that $v_j[A^T v]_j > 0 \forall A \in T(\bar{x})$. Chooses $s \in R^n$ so that $s_j = v_j$ and $s_i = 0$ for all $i \neq j$. Then $s^T A^T v = v_j [A^T v]_j > 0 \forall A \in T(\bar{x})$. Hence $\bar{x}$ is a strictly regular point.
As a consequence of the above theorem is the following corollary.

Corollary 3.1 *Let $f : R^n \rightarrow R^n$ be locally Lipschitzian. Let Φ be the square Fischer-Burmeister function. Suppose that $\Psi := \sum_{i=1}^{n} \Phi_i(\bar{x})$. Further assume that $\partial_B f(\bar{x})$ has the column-P_0-property. Then*

$$\Psi(\bar{x}) = 0 \Leftrightarrow 0 \in \partial \Psi(\bar{x}).$$

Proof. We note that by Corollary 1 in (Tawhid, Gowda, 2000), every matrix in $\partial f(\bar{x}) = co \ \partial_B f(\bar{x})$ is a P_0-matrix and by Corollary 2 in Tawhid, Goffin (2008), we have the claim.
Remark The usefulness of Corollary 3.1 may appear when the function f is piecewise smooth in which case $\partial_B f(\bar{x})$ consists of a finite number of matrices.

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Sintesi

I problemi di ottimizzazione svolgono un ruolo fondamentale nella determinazione di soluzioni per una grande varietà di questioni concrete ottenute tramite la modellizzazione di una data porzione di realtà, soprattutto quando alle soluzioni che si vogliono ottenere non si richiede di essere necessariamente esatte, quanto sostanzialmente buone. È evidente che la bontà di una soluzione dipende sia dal problema che si sta affrontando, sia dai parametri utilizzati per valutarla che possono essere rigorosi così come soggettivi; in tutti i casi, obbligatoriamente misurabili.

La matematica ha classicamente affrontato tali problematiche imponendo naturali criteri di regolarità alle diverse componenti coinvolte; una su tutte è la differenziabilità della funzione obiettivo. A partire da queste ipotesi le condizioni di ottimalità vengono calcolate anche tramite relazioni tra la derivate delle funzioni sotto esame. Tuttavia, per quanto questo approccio sia naturale, si è visto che questa supposta regolarità non è così scontata. In altre parole, nella descrizione della realtà entrano in gioco

fattori - l'intervento umano o le forze impulsive, ad esempio - che introducono salti nei fenomeni d'interesse, interrompendo il fluire continuo degli stessi: quest'ultimi quindi non possono essere più descritti dalla classe delle funzioni continue e tanto meno da quelle differenziabili.
Tali constatazioni hanno generato una nuova classe di problemi: i nonsmooth problem, nei quali le funzioni coinvolte non possiedono più necessariamente le caratteristica di essere derivabili.
Per quanto necessitino di un approccio più attento, numerosi risultati sono stati ottenuti all'interno delle ricerche su questa classe di problemi. Non ultimo tra questi è il contributo di Tawhid che, se da una parte approfondisce le connessioni presenti tra le soluzioni di casi di complementarietà non-lineare e i punti stazionari, locali e globali, di particolari funzioni obiettivo tra quelle di Fischer-Burmeister e di Kanzow-Kleinmichel, dall'altra consente di guardare sotto una nuova luce alcuni dei numerosi risultati a cui si è fatto riferimento, uniformandoli e estendendoli al contempo.
È anche questa unificazione a rendere innovativo il contributo: la dispersione dovuta a diversi approcci viene praticamente ridotta; ogni nuovo apporto interno alla prospettiva qui indicata porterà in cascata a miglioramenti in più campi: in tutti quelli le cui formalizzazioni ora fanno parte dell'unica e più ampia struttura resa visibile da Tawhid.