

# Fractal solids, product measures and fractional wave equations

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**ABSTRACT.** This paper builds on the recently begun extension of continuum thermomechanics to fractal porous media that are specified by a mass (or spatial) fractal dimension  $D$ , a surface fractal dimension  $d$  and a resolution length scale  $R$ . The focus is on pre-fractal media (i.e. those with lower and upper cut-offs) through a theory based on a dimensional regularization, in which  $D$  is also the order of fractional integrals employed to state global balance laws. In effect, the governing equations are cast in forms involving conventional (integer order) integrals, while the local forms are expressed through partial differential equations with derivatives of integer order but containing coefficients involving  $D$ ,  $d$  and  $R$ . This procedure allows a specification of a geometry configuration of continua by 'fractal metric' coefficients, on which the continuum mechanics is subsequently constructed. While all the derived relations depend explicitly on  $D$ ,  $d$  and  $R$ , upon setting  $D = 3$  and  $d = 2$ , they reduce to conventional forms of governing equations for continuous media with Euclidean geometries. Whereas the original formulation was based on a Riesz measure - and thus more suited to isotropic media - the new model is based on a product measure, making it capable of grasping local fractal anisotropy. Finally, the one-, two- and three-dimensional wave equations are developed, showing that the continuum mechanics approach is consistent with that obtained via variational energy principles.

**KEYWORDS:** Continuum mechanics, Fractals, Fractal derivative, Fractional integrals, Product measures, Waves

## I. Introduction

Fractals date back to research by Hausdorff and Besicovich on monster sets over 100 years ago, and then to the seminal work of Mandelbrot (1982). He was then followed by physicists and mathematicians. The first category was primarily condensed matter physicists who focused on the effects of fractal geometries of materials on bulk responses, e.g. Feder (1988). A number of specialized models have also been developed to particular problems like waves scattering at fractals (Berry, 1979), computational mechanics (Soare, Picu, 2007), fracture mechanics (Chudnovsky, Kunin, 1987; Balankin 1997; Carpinteri et al., 1999; Yavari et al., 2002a, b) or geomechanics (Dyskin, 2004). While in recent years mathematicians began to look at partial differential equations - starting with Laplace's or heat equation - on fractal (albeit non-random) sets (Kigami, 2001; Strichartz, 2006), an analogue of continuum physics and mechanics still needs to be developed. In particular, what is missing is a single unifying theoretical framework.

A new step in the direction of continuum physics and mechanics, relying on dimensional regularization, was taken by Tarasov (2005a, b, c). He developed continuum-type equations of conservation of mass, momentum and energy for fractal porous media, and on that basis studied several fluid mechanics and wave motion problems. In principle, one can then map a mechanics problem of a fractal [which is described by its mass ( $D$ ) and surface ( $d$ ) fractal dimensions plus the spatial resolution ( $R$ )] onto a problem in the Euclidean space in which this fractal is embedded, while having to deal with coefficients explicitly involving  $D$ ,  $d$  and  $R$ . As it turns out,  $D$  is also the order of fractional integrals employed to state global balance laws. Clearly, this has very interesting ramifications for formulating continuum-type mechanics of fractal media, which needs to be further explored. The great promise stems from the fact that the conventional requirement of continuum mechanics, the separation of scales, does not hold, yet the partial differential equations (with derivatives of integer order) may still be employed.

This paper builds on Tarasov's approach, which has already involved: an extension to continuum thermomechanics and thermoelasticity, a formulation of integral theorems, a generalization of extremum and variational principles and turbulence in fractal porous media (Ostoja-Starzewski, 2007a, 2007b, 2008, 2009a, 2009b). We first specify the geometry configuration of continua via 'fractal metric' coefficients, and on that basis construct a continuum mechanics.

Whereas the original formulation of Tarasov (and initially employed by us) was based on a Riesz measure - and thus more suited to isotropic fractals - the new model is based on a product measure, making it capable of grasping local fractal anisotropy. To verify our model, we require that the mechanical approach be consistent with that obtained from energy principles. To make the two approaches equivalent, we discuss the definition of fractal strain (with the fractal derivative developed) and verify product measures for anisotropic fractals. As an application, we study the one-dimensional, the two-dimensional anti-plane and the three-dimensional wave equations in general anisotropic fractal solids. While all the derived relations depend explicitly on  $D$ ,  $d$  and  $R$ , upon setting  $D = 3$  and  $d = 2$ , they reduce to conventional forms of governing equations for continuous media with Euclidean geometries.

## 2. Mass power law and fractional integrals

By a fractal solid, we understand a medium  $B$  having a fractal geometric structure. The mass of the medium  $m$  obeys a power law with respect to the lengthscale of measurement  $R$  (or resolution)

$$m(R) = kR^D, \quad D < 3, \quad (2.1)$$

where  $D$  is the fractal dimension of mass, and  $k$  is a proportionality constant. We note that, in practice, a fractional power law relation (2.1) is widely recognized and can be determined in experiments by a log-log plot of  $m$  and  $R$  (e.g. Schroeder, 1990), but the physical object is a pre-fractal, i.e. it has lower and upper cut-offs. Now, following Tarasov (2005a), the fractional integral is employed to represent mass in a three-dimensional region  $W$

$$m(W) = \int_W \rho(\mathbf{r}) dV_D = \int_W \rho(\mathbf{r}) c_3(D, r) dV_3. \quad (2.2)$$

Here the first and the second equality involve fractional integrals and conventional integrals, respectively. The coefficient  $c_3(D, r)$  provides a transformation between the two. Using Riesz fractional integrals,  $c_3(D, r)$  is

$$c_3(D, r) = \frac{2^{3-D} \Gamma(3/2)}{\Gamma(D/2)} r^{D-3}, \quad r = \sqrt{\sum_{i=1}^3 (x_i)^2}. \quad (2.3)$$

Note that  $c_3(D, r)$  above solely depends on the scalar distance  $r$ , which in turn confines the formulations to isotropic fractals. However, in general, the medium exhibits different fractal dimensions along different directions - it is anisotropic. A practical example of such a fractal anisotropy is given by Carpinteri et al. (1999, 2004), where a porous concrete structure is modelled by a Sierpiński carpet in the cross-section and a Cantor set along the longitudinal axis.

This consideration leads us to replace equation (2.1) by a more general power law relation with respect to each spatial coordinate

$$m(x_1, x_2, x_3) \sim x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}. \quad (2.4)$$

In order to account for such anisotropies, the fractional integral representing mass distribution is specified via a product measure

$$m(x_1, x_2, x_3) = \iiint \rho(x_1, x_2, x_3) d\mu_1(x_1) d\mu_2(x_2) d\mu_3(x_3). \quad (2.5)$$

Here the length measurement  $d\mu_k(x_k)$  in each coordinate is provided by

$$d\mu_k(x_k) = c_1^{(k)} (\alpha_k, x_k) dx_k, \quad k=1,2,3. \quad (2.6)$$

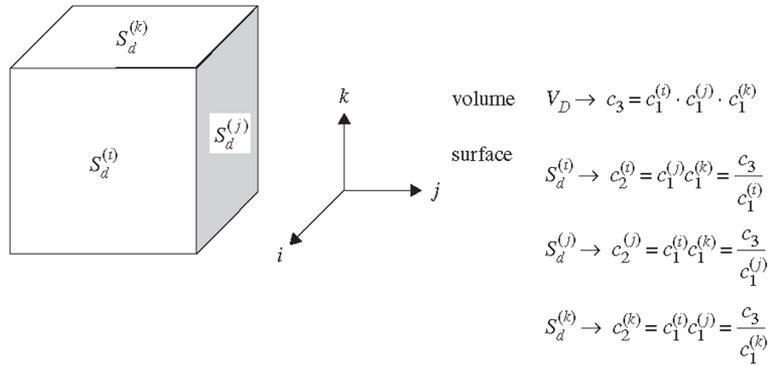
The total fractal dimension of mass  $D$  then equals  $\alpha_1 + \alpha_2 + \alpha_3$ . It follows that the volume coefficient  $c_3$  is given by

$$c_3 = c_1^{(1)} c_1^{(2)} c_1^{(3)} = \prod_{i=1}^3 c_1^{(i)}. \quad (2.7)$$

For the surface coefficient  $c_2$  we typically consider a cubic volume element, whose each surface element is specified by the corresponding normal vector (along the axis  $i, j$  or  $k$ , see Figure 1). Therefore, the coefficient  $c_2^{(k)}$  associated with the surface  $S_d(k)$  is

$$c_2^{(k)} = c_1^{(i)} c_1^{(j)} = \frac{c_3}{c_1^{(k)}}, \quad i \neq j \text{ and } i, j \neq k. \quad (2.8)$$

Figure 1. Constructing coefficients  $c_2$  and  $c_3$  via product measures



The expressions of length coefficients  $c_1^{(k)}$  depend on forms of specific fractional integrals. In the sequel, adopting a modified Riemann-Liouville fractional integral recently formulated by Jumarie (2005, 2008), we have

$$c_1^{(k)} = \alpha_k (l_k - x_k)^{\alpha_k - 1}, \quad k = 1, 2, 3, \quad (2.9)$$

where  $l_k$  is the total length (integral interval) along axis  $x_k$ . Let us now examine it in two special cases:

- 1) *Uniform mass.* The mass is distributed uniformly in a cubic region  $W$  with a power law relation (2.4). Denoting the mass density by  $\rho_0$  and the cubic length by  $l$ , we obtain

$$m(W) = \rho_0 l^{\alpha_1} l^{\alpha_2} l^{\alpha_3} = \rho_0 l^{\alpha_1 + \alpha_2 + \alpha_3} = \rho_0 l^D, \quad (2.10)$$

which is consistent with the mass power law (2.1).

- 2) *Point mass.* The distribution of mass is concentrated at one point, so that the mass density is denoted by the Dirac function  $\rho(x_1, x_2, x_3) = m_0 \delta(x_1) \delta(x_2) \delta(x_3)$ . The fractional integral representing mass becomes

$$m(W) = \alpha_1 \alpha_2 \alpha_3 l^{\alpha_1 - 1} l^{\alpha_2 - 1} l^{\alpha_3 - 1} m_0 = \alpha_1 \alpha_2 \alpha_3 l^{D-3} m_0. \quad (2.11)$$

When  $D \rightarrow 3 (\alpha_1, \alpha_2, \alpha_3 \rightarrow 1)$ ,  $m(W) \rightarrow m_0$  and the conventional concept of point mass is recovered (Temam, Miranville, 2005). Note that using the Riesz fractional integral will always give zero ( $0^{D-3}$ ) except when  $D = 3$  (by letting  $0^0 = 1$ ), which,

on the other hand, shows a non-smooth transition of mass with respect to its fractal dimension. This also supports our choice of the non-Riesz-type expressions for  $c_1^{(k)}$  in equation (2.9).

Note that the above expression  $c_1^{(k)}$  shows that the length dimension, and hence the mass  $m$ , will involve an unusual physical dimension following from the fractional integral (2.5). This is understandable since in mathematics a fractal curve exhibits only a finite measure with respect to a fractal dimensional length unit (Mandelbrot, 1982). Since, in practice, we prefer physical quantities to have usual dimensions, an alternative way to address this issue is to non-dimensionalize the coefficients  $c_1^{(k)}$ . Therefore, we suggest replacing  $(l_k - x_k)$  by  $(l_k - x_k) / l_0$  in equation (2.9) ( $l_0$  is a characteristic scale, e.g. the mean pore size).

### 3. Fractional integral theorems and fractal derivatives

At this point, we recall two basic integral theorems extensively employed in continuum mechanics: Gauss's theorem, which relates a certain volume integral to the integral over its bounding surface, and the Reynolds transport theorem, concerning the rate of change of any volume integral for a continuous medium. In the following, we derive their fractional generalizations and, moreover, introduce a definition of fractal derivatives, which together provide a stepping-stone to construct a continuum mechanics in the setting of fractals. The derivation of a fractional Gauss's theorem will be analogous to Tarasov's (2005b) dimensional regularization, albeit formulated in the framework of product measures discussed above. First, let us recall the surface integral in a fractal medium

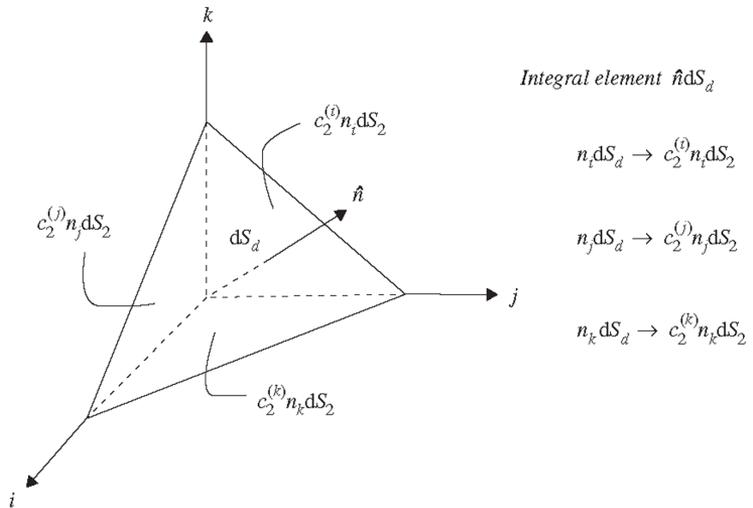
$$S_d[\bar{f}] := \int_{S_d} \bar{f} \cdot \hat{n} dS_d = \int_{S_d} f_k n_k dS_d . \quad (3.1)$$

Here  $\bar{f} = f_k e_k$  is any vector field and  $\hat{n} = n_k e_k$  is the unit normal vector of the surface; the Einstein summation convention is assumed. In order to compute equation (3.1), we relate the integral element  $\hat{n} dS_d$  to its conventional forms  $\hat{n} dS_2$  via fractal surface coefficients  $c_2^{(i)}$ ,  $c_2^{(j)}$ ,  $c_2^{(k)}$ . Note that, by definition, any infinitesimal surface element  $dS_d$  in the integrand can be regarded as a plane (aligned in an arbitrary direction with a normal vector  $\hat{n}$ ). Since the coefficients

$c_2^{(i)}$ 's are built on coordinate planes  $Ox_j, x_k$ 's, we consider their projections onto each coordinate plane. The projected planes  $n_i dS_d$  can then be specified by coefficients  $c_2^{(i)}$ 's, and this totally provides a representation of the integral element  $\hat{n}dS_d$  (Figure 2). Thus, we have

$$\int_{S_d} \bar{f} \cdot \hat{n}dS_d = \int_{S_2} f_k c_2^{(k)} n_k dS_2 \tag{3.2}$$

Figure 2. A representation of the fractional integral element  $\hat{n}dS_d$  under product measures



Now, following the conventional Gauss's theorem, we get

$$\int_{\partial W} f_k c_2^{(k)} n_k dS_2 = \int_W (f_k c_2^{(k)})_{,k} dV_3 \tag{3.3}$$

Note that from the expression (2.8)  $c_2^{(k)}$  is independent of the variable  $x_k$ , so that we can write equation (3.3) as

$$\int_{\partial W} f_k n_k dS_d = \int_W (f_k c_2^{(k)})_{,k} c_3^{-1} dV_D = \int_W f_{k,k} c_2^{(k)} c_3^{-1} dV_D = \int_W \frac{f_{k,k}}{c_1^{(k)}} dV_D := \int_W \nabla_k^D f_k dV_D \tag{3.4}$$

This equation is a fractional generalization of Gauss's theorem. Hereafter, we use the notation of fractal derivative,  $\nabla_k^D$ , with respect to the coordinate  $x_k$

$$\nabla_k^D := \frac{1}{c_1^{(k)}} \frac{\partial}{\partial x_k} (\cdot) \tag{3.5}$$

The definition of  $\nabla_k^D$  is similar to Tarasov's (2005b)  $\nabla_k^D = c_3^{-1}(c_2 \cdot)_{,k}$ , albeit our form is simplified for product measures. We now examine three properties of the operator  $\nabla_k^D$ .

1) *It is the 'inverse' operator of fractional integrals.* For any function  $f(x)$ , we have

$$\nabla_x^D \int f(x) d\mu^D(x) = \frac{1}{c_1(x)} \frac{d}{dx} \int f(x) c_1(x) dx = \frac{1}{c_1(x)} [f(x) c_1(x)] = f(x) \quad (3.6)$$

and

$$\int \nabla_x^D f(x) d\mu^D(x) = \int \left[ \frac{1}{c_1(x)} \frac{df(x)}{dx} \right] c_1(x) dx = \int \frac{df(x)}{dx} dx = f(x). \quad (3.7)$$

For this reason, we call  $\nabla_k^D$  a 'fractal derivative' (so as to distinguish it from the fractional derivatives already in existence).

2) *The rule of 'term-by-term' differentiation is satisfied*

$$\nabla_k^D (AB) = \frac{1}{c_1^{(k)}} \frac{\partial}{\partial x_k} (AB) = \frac{1}{c_1^{(k)}} \frac{\partial(A)}{\partial x_k} B + \frac{1}{c_1^{(k)}} \frac{\partial(B)}{\partial x_k} A = B \nabla_k^D (A) + A \nabla_k^D (B), \quad (3.8)$$

whereby we note that this is invalid in Tarasov's (2005b) notation.

3) *Its operation on any constant is zero*

$$\nabla_k^D (C) = \frac{1}{c_1^{(k)}} \frac{\partial(C)}{\partial x_k} = 0. \quad (3.9)$$

Here we recall that the usual fractional derivative (Riemann-Liouville) of a constant does not equal zero - neither in fractional calculus (Oldham, Spanier, 1974) nor in Tarasov's (2005b) formulation.

As to the fractional generalization of Reynolds transport theorem, we follow the line of conventional continuum mechanics distinguishing between the reference and deformed configurations

Figure 3. Typical RLO  
(Thinking about Thinking)

$$\begin{aligned}
\frac{d}{dt} \int_{W_t} P dV_D &= \frac{d}{dt} \int_{W_0} P J dV_D^0 = \int_{W_0} \frac{d}{dt} (PJ) dV_D^0 = \int_{W_0} \left( \frac{d}{dt} P \cdot J + P \cdot \frac{d}{dt} J \right) dV_D^0 \\
&= \int_{W_0} \left( \frac{d}{dt} P \cdot J + P \cdot v_{k,k} J \right) dV_D^0 = \int_{W_0} \left( \frac{d}{dt} P + P \cdot v_{k,k} \right) J dV_D^0 \\
&= \int_{W_t} \left( \frac{d}{dt} P + P \cdot v_{k,k} \right) dV_D = \int_{W_t} \left( \frac{\partial}{\partial t} P + P_{,k} v_k + P \cdot v_{k,k} \right) dV_D \\
&= \int_{W_t} \left( \frac{\partial}{\partial t} P + (P v_k)_{,k} \right) dV_D.
\end{aligned} \tag{3.10}$$

Here  $P$  is any quantity accompanied by a moving material system  $W_t$ ,  $v = v_k e_k$  is the velocity field and  $J$  is the Jacobian of the transformation from the current configuration  $x_k$  to the referential configuration  $X_K$ . Note that the result is identical to its conventional representation. The fractal material time derivative is thus the same

$$\left( \frac{d}{dt} \right)_D P = \frac{d}{dt} P = \frac{\partial}{\partial t} P + P_{,k} v_k, \tag{3.11}$$

whereby we note that the alternative form of the fractional Reynolds transport theorem which involves surface integrals is different from the conventional one and rather complicated. This is because the fractal volume coefficient  $c_3$  depends on all coordinates  $x_k$ 's (not like  $c_2^{(k)}$ , which is independent of  $x_k$  when deriving fractional Gauss's theorem). Continuing with equation (3.10), the derivation follows as

$$\begin{aligned}
\frac{d}{dt} \int_{W_t} P dV_D &= \int_{W_t} \left( \frac{\partial}{\partial t} P + (P v_k)_{,k} \right) dV_D = \int_{W_t} \frac{\partial}{\partial t} P dV_D + \int_{W_t} (P v_k)_{,k} c_3 dV_3 \\
&= \int_{W_t} \frac{\partial}{\partial t} P dV_D + \int_{W_t} \left( \int (P v_k)_{,k} c_3 dx_k \right)_{,k} dV_3 = \int_{W_t} \frac{\partial}{\partial t} P dV_D + \int_{\partial W_t} \left( \int (P v_k)_{,k} c_3 dx_k \right) n_k dS_2 \\
&= \int_{W_t} \frac{\partial}{\partial t} P dV_D + \int_{\partial W_t} \left( P v_k c_3 - \int P v_k c_{3,k} dx_k \right) (c_2^{(k)})^{-1} n_k dS_d \\
&= \int_{W_t} \frac{\partial}{\partial t} P dV_D + \int_{\partial W_t} P c_1^{(k)} v_k n_k dS_d - \int_{\partial W_t} \left( \int P c_{1,k}^{(k)} v_k dx_k \right) n_k dS_d.
\end{aligned} \tag{3.12}$$

#### 4. Continuum mechanics of fractal solids

In the second and third sections we have discussed fractional integrals under product measures and thereby generalized some basic integral theorems. Now, we proceed to develop a framework of continuum mechanics in fractal setting. We will formulate the field equations analogous to those in classical continuum mechanics but based on fractional integrals. Note that the notions of continuum mechanics rely on geometry configurations of the body. We shall first examine some

physical concepts and definitions on account of the fractal geometry. Let us recall the formula of fractal mass (equation 2.2) which expresses the mass power law via fractional integrals. From a homogenization standpoint, this allows an interpretation of the fractal (intrinsically discontinuous or irregular) medium as a continuum and a 'fractal metric' embedded in the equivalent 'homogenized' continuum model, saying that

$$dl_D = c_1 dx, \quad dS_d = c_2 dS_2, \quad dV_D = c_3 dV_3. \quad (4.1)$$

Here  $dl_D$ ,  $dS_d$ ,  $dV_D$  represent the line, surface, volume element in the fractal body and  $dx$ ,  $dS_2$ ,  $dV_3$  denote those in the homogenized model, see Figure 3. The coefficients  $c_1$ ,  $c_2$ ,  $c_3$  provide the relation between both sets.

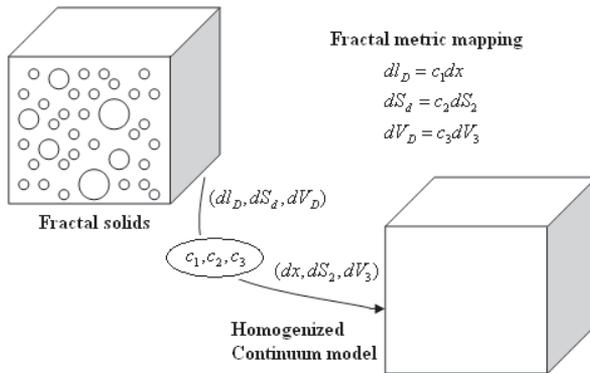


Figure 3. An illustration of the homogenization process from geometry configurations

The definitions of stress and strain must also be modified accordingly. The Cauchy stress is now specified to express the surface force  $F_k^S$  via fractional integrals

$$F_k^S = \int_{\partial W} \sigma_{ki} n_i dS_d = \int_{\partial W} \sigma_{ki} n_i c_2^{(i)} dS_2 \quad (4.2)$$

As to the configuration of strain, we recommend replacing all the spatial derivatives  $\partial / \partial x_k$  with fractal derivatives  $\nabla_k^D$  introduced in section 3. This can be understood by observing from equation (4.1) that

$$\nabla_k^D = \frac{1}{c_1^{(k)}} \frac{\partial}{\partial x_k} = \frac{\partial}{\partial l_D^{(k)}}. \quad (4.3)$$

For small deformation, the expression of strain in fractal solids thus gives

$$\varepsilon_{ij} = \frac{1}{2}(\nabla_j^D u_i + \nabla_i^D u_j) = \frac{1}{2}\left(\frac{1}{c_1^{(j)}} u_{i,j} + \frac{1}{c_1^{(i)}} u_{j,i}\right). \quad (4.4)$$

Note that the stress-strain pairs must be conjugate from the viewpoint of energy. We shall examine the consistency of these definitions later when deriving wave equations in the next section. At this point, let us consider the balance law of linear momentum in fractal solids

$$\frac{d}{dt} \int_W \rho \mathbf{v} dV_D = \mathbf{F}^B + \mathbf{F}^S, \quad (4.5)$$

where  $\mathbf{v} = v_k \mathbf{e}_k$  denotes the velocity vector, and  $\mathbf{F}^B$ ,  $\mathbf{F}^S$  are the body and surface forces, respectively. Writing equation (4.5) in indicial notation and expressing forces in terms of fractional integrals yield

$$\frac{d}{dt} \int_W \rho v_k dV_D = \int_W f_k dV_D + \int_{\partial W} \sigma_{kl} n_l dS_d. \quad (4.6)$$

On observation of fractional Gauss's theorem (3.4) and Reynolds transport theorem (3.10), this gives

$$\int_W \rho \left( \frac{d}{dt} \right)_D v_k dV_D = \int_W (f_k + \nabla_l^D \sigma_{kl}) dV_D. \quad (4.7)$$

Here the operators of fractal derivative  $\nabla_k^D$  and material derivative  $\left( \frac{d}{dt} \right)_D$  are employed; recall equations (3.5) and (3.11).

Note that the region  $W$  is arbitrary. On account of equation (4.7), we obtain the balance equation in local form

$$\rho \left( \frac{d}{dt} \right)_D v_k = f_k + \nabla_l^D \sigma_{kl}. \quad (4.8)$$

The consideration of angular momentum leads to micropolar effects (Li, Ostoja-Starzewski, 2011).

The specification of constitutive equations involves more arguments from physics, and we recommend keeping the relations of stress and strain while modifying their definitions to fractal setting. This is understood in that the fractal geometry solely influences our

configurations of some physical quantities (like stress and strain) while it takes no effect on physical laws (like the conservation principles, and constitutive relations that are inherently due to material properties). We note that this justification is verified in Carpinteri and Pugno (2005) where scale effects of material strength and stress are discussed from the standpoint of fractal geometry and confirmed by experiments of both brittle and plastic materials.

Now, we consider a specific example: linear elastic solids with small deformation. The constitutive equations take the usual linear forms

$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij} , \quad (4.9)$$

where  $\lambda$  and  $\mu$  are material parameters (Lamé constants), while  $\sigma_{ij}$  and  $\varepsilon_{ij}$  are fractal stress and strain defined in equations (4.2) and (4.4), respectively.

Under small displacements, the linearization of equation (4.8) gives

$$\rho \frac{\partial^2 u_k}{\partial t^2} = f_k + \nabla_i^D \sigma_{ki} , \quad (4.10)$$

where  $\mathbf{u} = u_k \mathbf{e}_k$  is the displacement field. Note that equations (4.4), (4.9) and (4.10) constitute a complete set of equations describing the problem (excluding boundary conditions).

## 5. Fractional wave equations

It is now possible to study wave motion in fractal solids based on the continuum-type equations derived in section 4. As a starting point here, we will exclusively consider waves in linear elastic homogeneous fractal solids under small motions and zero external loads. Equations (4.4), (4.9) and (4.10) can thus jointly lead to wave equations. Note that variational principles provide an alternative approach to studying elastic problems. We shall therefore derive fractional wave equations via these two approaches and examine whether the results are consistent. This can be regarded as a verification of our entire formulation.

### (a) One-dimensional plane wave

The 1D plane wave motion involves one spatial variable only, say  $x_1$  or just  $x$ . We will consider the derivations under mechanical and variational

approaches, respectively. It is examined in the simplest case whether our definitions of fractal stress and strain in section 4 are self-consistent.

### 1) Mechanical approach

The balance of linear momentum reduces to:

$$\rho \ddot{u} = c_1^{-1} \sigma_{,x} . \quad (5.1)$$

The constitutive equation becomes:

$$\sigma = E \varepsilon , \quad (5.2)$$

where we recognize Young's modulus  $E$ . Substituting equation (5.2) into equation (5.1), we obtain

$$\rho \ddot{u} = E c_1^{-1} \varepsilon_{,x} . \quad (5.3)$$

Note that the strain  $\varepsilon$  is defined as a function of the displacement  $u$  (usually the derivative). The wave equation can then be derived from equation (5.3). Following the conventional strain definition,  $\varepsilon = u_{,x}$ , which substituted into equation (5.3) gives

$$\rho \ddot{u} = E c_1^{-1} u_{,xx} . \quad (5.4a)$$

On the other hand, using our definition (4.4), simplified to 1D,  $\varepsilon = c_1^{-1} u_{,x}$ , which yields

$$\rho \ddot{u} = E c_1^{-1} (c_1^{-1} u_{,x})_{,x} . \quad (5.4b)$$

### 2) Variational approach

In the variational approach, we consider the kinetic energy  $T$  and the strain energy  $U$  associated with the medium. The wave equation follows from Hamilton's principle that acts on its Lagrangian function  $L = T - U$ . First, the kinetic energy is

$$T = \frac{1}{2} \rho \int \dot{u}^2 dl_D = \frac{1}{2} \rho \int \dot{u}^2 c_1 dx , \quad (5.5)$$

while the strain energy is

$$U = \frac{1}{2} E \int \varepsilon^2 dl_D = \frac{1}{2} E \int \varepsilon^2 c_1 dx . \quad (5.6)$$

Employing the conventional definition of strain, equation (5.6) becomes

$$U = \frac{1}{2} E \int c_1 u_{,x}^2 dx , \quad (5.7a)$$

while using our fractal definition of strain gives

$$U = \frac{1}{2} E \int c_1^{-1} u_{,x}^2 dx . \quad (5.7b)$$

According to Hamilton's principle,  $\delta \int L dt = \delta \int (T - U) dt = 0$ , which implies the Euler-Lagrange equation

$$\frac{\partial}{\partial t} \left[ \frac{\partial \ell}{\partial \dot{u}} \right] + \frac{\partial}{\partial x} \left[ \frac{\partial \ell}{\partial u_{,x}} \right] - \frac{\partial \ell}{\partial u} = 0 , \quad (5.8)$$

where  $\ell$  denotes the Lagrangian density, defined by  $L = \int \ell dx$ . For the conventional definition of strain,  $\ell$  has the form

$$\ell = \frac{1}{2} (\rho c_1 \dot{u}^2 - E c_1 u_{,x}^2) , \quad (5.9a)$$

while the fractal definition of strain gives

$$\ell = \frac{1}{2} (\rho c_1 \dot{u}^2 - E c_1^{-1} u_{,x}^2) . \quad (5.9b)$$

Substituting equation (5.9a) or (5.9b) into equation (5.8), we obtain, respectively, the wave equations

$$\rho c_1 \ddot{u} - E (c_1 u_{,x})_{,x} = 0 , \quad (5.10a)$$

$$\rho c_1 \ddot{u} - E (c_1^{-1} u_{,x})_{,x} = 0 . \quad (5.10b)$$

Comparing the results between the mechanical and variational approaches, we find that equation (5.4b) agrees with (5.10b), while equation (5.4a) contradicts (5.10a) (Tarasov, 2005b, c). Thus, our definitions of fractal stress and strain are self-consistent.

An analogous result, also exhibiting self-consistency, has recently been obtained in elastodynamics of fractally structured Timoshenko beams (Ostoja-Starzewski, Li, 2009) and was extended to finite motions in

(Demie, Ostoja-Starzewski, 2011).

**(b) Two-dimensional anti-plane wave**

A two-dimensional anti-plane wave is described by a displacement field  $u_3(x_1, x_2, t)$  ( $u_1$  and  $u_2$  vanishes). As before, we study it under mechanical and variational approaches. This then provides a simple case to examine our construction of the surface coefficient  $c_2$  via product measures in section 2. For the local balance of linear momentum, only the one involving  $u_3$  is of interest

$$\rho \ddot{u}_3 = \nabla_k^D \sigma_{3k,k} . \quad (5.11)$$

Note that the mean strain  $\varepsilon_{kk}$  is zero, and so the constitutive equations reduce to

$$\sigma_{ij} = 2\mu \varepsilon_{ij} . \quad (5.12)$$

The corresponding stress components in equation (5.11) follow from equations (5.12) and (4.4) as

$$\sigma_{3k} = \mu \nabla_k^D u_3 = \mu \frac{u_{3,k}}{c_1^{(k)}} . \quad (5.13)$$

Substituting equation (5.13) into equation (5.11), we obtain the wave equation

$$\rho \ddot{u}_3 = \mu \left[ \frac{1}{c_1^{(1)}} \left( \frac{u_{3,1}}{c_1^{(1)}} \right)_{,1} + \frac{1}{c_1^{(2)}} \left( \frac{u_{3,2}}{c_1^{(2)}} \right)_{,2} \right] . \quad (5.14)$$

As to the variational approach, we consider the body with a unit length in  $x_3$ .

The kinetic energy thus gives (since only surface coefficient  $c_2^{(3)}$  is involved, for simplicity we denote it as  $c_2$ )

$$T = \frac{1}{2} \rho \iint \dot{u}_3^2 dS_d = \frac{1}{2} \rho \iint \dot{u}_3^2 c_2 dS_2 . \quad (5.15)$$

The strain energy is

$$U = \frac{1}{2} \iint \sigma_{ij} \varepsilon_{ij} dS_d = \frac{1}{2} \mu \iint (\nabla_k^D u_3) (\nabla_k^D u_3) c_2 dS_2 . \quad (5.16)$$

The Lagrangian density has the form

$$\begin{aligned} \ell &= \frac{1}{2} \rho c_2 \dot{u}_3^2 - \frac{1}{2} \mu c_2 (\nabla_k^D u_3) (\nabla_k^D u_3) = \frac{1}{2} \rho c_1^{(1)} c_1^{(2)} \dot{u}_3^2 - \frac{1}{2} \mu c_1^{(1)} c_1^{(2)} \left[ \left( \frac{u_{3,1}}{c_1^{(1)}} \right)^2 + \left( \frac{u_{3,2}}{c_1^{(2)}} \right)^2 \right] \\ &= \frac{1}{2} \rho c_1^{(1)} c_1^{(2)} \dot{u}_3^2 - \frac{1}{2} \mu \left[ \frac{c_1^{(2)}}{c_1^{(1)}} u_{3,1}^2 + \frac{c_1^{(1)}}{c_1^{(2)}} u_{3,2}^2 \right]. \end{aligned} \quad (5.17)$$

Next, applying the Euler-Lagrange equation

$$\frac{\partial}{\partial t} \left[ \frac{\partial \ell}{\partial \dot{u}_3} \right] + \sum_{k=1}^2 \frac{\partial}{\partial x_k} \left[ \frac{\partial \ell}{\partial u_{3,k}} \right] - \frac{\partial \ell}{\partial u_3} = 0, \quad (5.18)$$

we derive a wave equation

$$\rho c_1^{(1)} c_1^{(2)} \ddot{u}_3 - \mu \left[ c_1^{(2)} \left( \frac{u_{3,1}}{c_1^{(1)}} \right)_{,1} + c_1^{(1)} \left( \frac{u_{3,2}}{c_1^{(2)}} \right)_{,2} \right] = 0, \quad (5.19)$$

and note that it is equivalent to equation (5.14). This verifies our expression for  $c_2$  via product measures in equation (2.8).

### (c) Three-dimensional waves

We now proceed to discuss the most general case: three-dimensional waves that involve all spatial variables:  $x_1, x_2, x_3$ . Similar to the above derivations, in the mechanical approach we eliminate the stresses by displacements via strain displacement relations (4.4) and constitutive laws (4.9), and then arrive at the wave equation from equation (4.10). The results are a little more complicated and have the form

$$\rho \ddot{u}_i = \mu \nabla_j^D \nabla_j^D u_i + (\lambda + \mu) \nabla_i^D \nabla_j^D u_j,$$

i.e.

$$\rho \ddot{u}_i = \mu \frac{1}{c_1^{(j)}} \left( \frac{u_{i,j}}{c_1^{(j)}} \right)_{,j} + (\lambda + \mu) \frac{1}{c_1^{(i)}} \left( \frac{u_{j,j}}{c_1^{(j)}} \right)_{,i}. \quad (5.20)$$

On the other hand, in the variational approach the Lagrange density follows as

$$\begin{aligned} \ell &= \frac{1}{2} \rho c_3 \dot{u}_i \dot{u}_i - \frac{1}{2} c_3 \sigma_{ij} \varepsilon_{ij} = \frac{1}{2} \rho c_3 \dot{u}_i \dot{u}_i - \left[ \frac{1}{2} \lambda c_3 (\varepsilon_{kk})^2 + \mu c_3 \varepsilon_{ij} \varepsilon_{ij} \right] \\ &= \frac{1}{2} \rho c_3 \dot{u}_i \dot{u}_i - \frac{1}{2} c_3 \left[ \lambda (\nabla_k^D u_k)^2 + \mu (\nabla_j^D u_i \nabla_j^D u_i + \nabla_j^D u_i \nabla_i^D u_j) \right] \\ &= \frac{1}{2} \rho c_3 \dot{u}_i \dot{u}_i - \frac{1}{2} c_3 \left[ \lambda \left( \frac{u_{k,k}}{c_1^{(k)}} \right)^2 + \mu \left( \left( \frac{u_{i,j}}{c_1^{(j)}} \right) \left( \frac{u_{i,j}}{c_1^{(j)}} \right) + \left( \frac{u_{i,j}}{c_1^{(j)}} \right) \left( \frac{u_{j,i}}{c_1^{(i)}} \right) \right) \right]. \end{aligned} \quad (5.21)$$

The Euler-Lagrange equations are given by

$$\frac{\partial}{\partial t} \left[ \frac{\partial \ell}{\partial \dot{u}_i} \right] + \sum_{j=1}^3 \frac{\partial}{\partial x_j} \left[ \frac{\partial \ell}{\partial u_{i,j}} \right] - \frac{\partial \ell}{\partial u_i} = 0. \quad (5.22)$$

Substituting equation (5.21) into (5.22), we find

$$\rho c_3 \ddot{u}_i - \mu \left( \frac{c_3 u_{i,j}}{(c_1^{(j)})^2} \right)_{,j} - (\lambda + \mu) \left( \frac{c_3 u_{j,i}}{c_1^{(i)} c_1^{(j)}} \right)_{,j} = 0. \quad (5.23)$$

On account of the formulation of product measures, we have

$$\frac{c_3}{(c_1^{(j)})^2} = \frac{c_2^{(j)} c_1^{(j)}}{(c_1^{(j)})^2} = \frac{c_2^{(j)}}{c_1^{(j)}} \quad \text{and} \quad \frac{c_3}{c_1^{(i)} c_1^{(j)}} = \frac{c_1^{(i)} c_2^{(i)}}{c_1^{(i)} c_1^{(j)}} = \frac{c_2^{(i)}}{c_1^{(j)}} \quad (5.24)$$

Note that the expression  $\left( \frac{c_3 u_{j,i}}{c_1^{(i)} c_1^{(j)}} \right)_{,j} = \left( \frac{c_2 u_{j,j}}{c_1^{(i)} c_1^{(j)}} \right)_{,j}$  involves a summation

over  $j = 1, 2, 3$  and according to equation (2.8)  $c_2^{(j)}$  is independent of  $x_j$  while  $c_2^{(i)}$  is independent of  $x_i$ , so that, effectively, equation (5.23) becomes (5.20). On the other hand, we note that in Tarasov's (2005a, b) expressions for  $c_1, \dots, c_3$  - where Riesz fractional integrals were adopted - the forms of fractional wave equations are more complicated and they are not equivalent under these two approaches. This and other comments in this paper are not meant as a criticism of Tarasov's work as, indeed, we have been very much motivated by his research.

## 6. Conclusions

We presented a self-consistent approach to mechanics of continua with fractal geometries. The fractal characteristics are captured by fractional integrals following from the mass power law of the body. On this basis a framework of continuum mechanics of fractal media is developed and a particular example concerning one-dimensional, two-dimensional anti-plane and three-dimensional waves in linear elastic fractal solids is studied, showing that the result of the mechanical approach is equivalent to that obtained by variational principles, thus verifying the self-consistency of our theoretical framework. Finally, our method suggests specific expressions for  $c_1, \dots, c_3$  and, therefore, can be easily applied in practice when the power law of fractal mass is measured.

We note that the framework remains self-consistent whenever the expressions for  $c_1, \dots, c_3$  are constructed via product measures and such a property is independent of a specific form of  $c_1$ . This allows the modification of  $c_1$  to possibly incorporate the asymptotic behaviours in lower and upper cut-offs of physical pre-fractals. Although our current focus is on fractal solids, the work can be potentially extended to fluid mechanics like the shear flow turbulence, e.g. regarding the anisotropic drag reducing flow (McComb, Chan 1979, 1981).

As a final note, the coefficients  $c_1^{(i)}, c_1^{(j)}, c_1^{(k)}$  and  $c_2^{(i)}, c_2^{(j)}, c_2^{(k)}$  are not involved in the summation convention. For example, (3.2) is written explicitly as

$$\int_{S_d} \bar{f} \cdot \hat{n} dS_d = \int_{S_2} f_k c_2^{(k)} n_k dS_2 = \sum_{k=1}^3 \int_{S_2} f_k c_2^{(k)} n_k dS_2 \quad (6.1)$$

owing to the repeated index in  $f_k$  and  $n_k$ . Also note that equation (3.5) does not involve a summation on  $k$ , although  $c_1^{(k)}$  and  $x_k$  have a repeated index 'k'

$$\nabla_k^D := \frac{1}{c_1^{(k)}} \frac{\partial}{\partial x_k} (\cdot) \neq \sum_{k=1}^3 \frac{1}{c_1^{(k)}} \frac{\partial}{\partial x_k} (\cdot) \quad (6.2)$$

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## Sintesi

A Euclide, matematico greco del IV-III sec. a.C., si deve una delle opere più influenti della storia della matematica: gli Elementi. Probabile discepolo di Platone, bibliotecario di Alessandria, coevo di Eratostene e Archimede, con la sua opera da una parte riordinò le conoscenze geometriche della sua epoca, secondo la struttura ipotetico-deduttiva tuttora in uso nella didattica scolastica; dall'altra si spinse oltre il sapere contemporaneo, ottenendo risultati di grande rilevanza e bellezza. La forza degli Elementi tuttavia non è soltanto nei risultati, ma soprattutto nell'esposizione e nello svincolarsi dalle nozioni dettate dal senso comune e soggettive, integrando queste nella struttura stessa dell'opera, ponendole come vere o false in modo definitivo.

Negli Elementi, Euclide pone l'idea di punto, di linea, di superficie: "punto è ciò che non ha parti" (Termini, I), "linea è lunghezza senza larghezza" (Termini, II) e "superficie è ciò che ha soltanto lunghezza e larghezza" (Termini, V). Sottintese a queste c'è l'idea di dimensione, ovvero della proprietà di un oggetto matematico di estendersi nello spazio: il punto non si estende, la linea lo fa in una sola direzione, la superficie in due.

Per duemila anni quest'idea naturale è rimasta uguale a se stessa, non essendoci stato nessun risultato matematico, né alcuna nuova teoria fisica a stimolarne un'evoluzione. Tuttavia, nel 1918, Hausdorff ampliò l'idea euclidea di dimensione introducendo la dimensione frattale o dimensione di Hausdorff. Fu un cambio del punto di vista: la dimensione di un oggetto non è tanto una misura delle direzioni rispetto alle quali esso si estende, quanto la misura delle direzioni occupate all'interno dello spazio in cui è immerso.

Possiamo evidenziare la differenza usando l'esempio di un foglio di carta accartocciato: di fronte a questo Euclide è in en passe, avendo a che fare con un oggetto piano che si estende in tre direzioni. Per Hausdorff, il foglio si sviluppa in tre direzioni, anche se non le occupa pienamente. La dimensione di Hausdorff di conseguenza risulterà maggiore di 2 e minore di 3 (il valore esatto dipende dal modo in cui è stato accartocciato): più del foglio, meno dello spazio.

Nata come idea puramente matematica, la dimensione frattale ha avuto nel 1967 uno sviluppo inatteso e dirompente dopo che il polacco, naturalizzato francese, Mandelbrot la utilizzò per il calcolo della lunghezza delle coste britanniche. Da quel momento la visione frattale della natura ha permeato le scienze naturali.

Per sviluppare questa nuova visione è stato necessario introdurre nuovi strumenti matematici. Tornando all'esempio del foglio, in Euclide è sufficiente un righello, in Hausdorff questo potrebbe non bastare poiché fornirebbe un'approssimazione lontana dal vero lì dove sono presenti le pieghe. Il righello, qui, è un esempio tangibile per descrivere ciò che è successo nell'ambito prettamente matematico. Nelel scienze

matematiche, infatti, strumenti classici, quali derivazione e integrazione, sono stati ampliati per poterli adattare alla nuova idea di dimensione. Tale procedimento ha portato alle derivate di ordine non intero e all'integrazione su sequenze di domini che si adattano ricorsivamente alla realtà frattale delle entità oggetto di misura.

In questo contesto, si inserisce il lavoro di Li e Ostoj-Starzewski, che, facendo seguito ai recenti progressi ottenuti, tra gli altri, da Tarasov, Kigami e Strichartz, mette in luce un nuovo efficace impiego di questi strumenti matematici. Un esempio su tutti: l'utilizzo, per la misura di volumi in cui sono presenti bolle a ogni ordine di grandezza, dell'integrazione di combinazioni lineari di derivate parziali di ordine intero, i cui coefficienti sono legati alla dimensione frattale dell'oggetto, a quella della sua superficie, e alla lunghezza di scala. È da sottolineare che la validità dei risultati conseguiti è comprovata dal fatto che, quando il metodo è applicato a problemi risolti in precedenza, ciò che si ottiene converge verso quanto già noto.

Ramo giovane della matematica, l'analisi frattale si propone come valido strumento per lo studio di quella zona al confine tra il lineare e il caotico, lì dove il velo che separa il mondo macroscopico e quello microscopico si fa sottile fino a scomparire.